2019-Nov-19 (pre-lecture)

MATB44 Prof. Yu William Yu

Monday, November 18, 2019 1:15 PM

Outline:

- Linearization and stability analysis
- Difference equations

Previously:

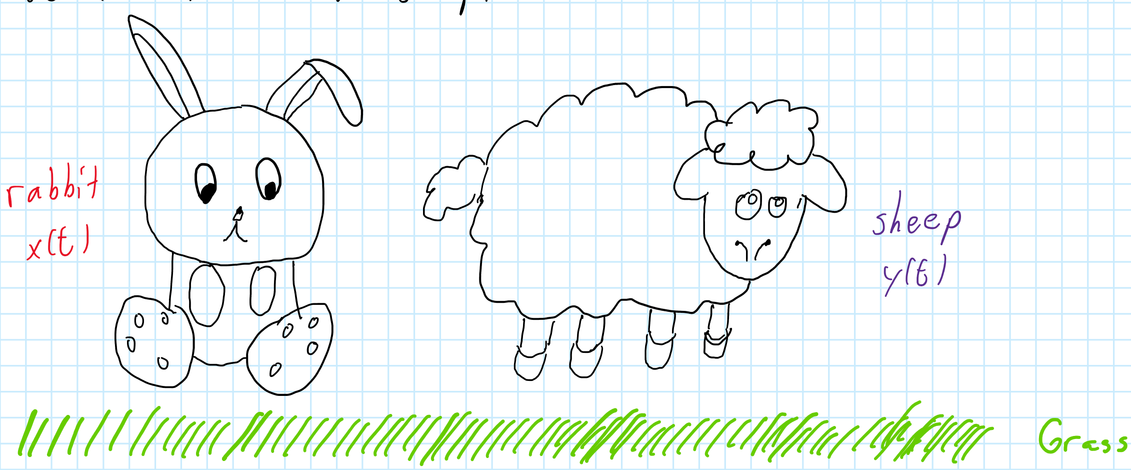
We classified linear 2D systems. $\dot{x} = Ax$

i.e. if the real parts of both eigenvalues are negative, then the system is stable at the origin, and is a sink.

Can we do something similar with nonlinear and/or nonhomogeneous systems?

Lotka-Volterra

Suppose we have rabbits & sheep.



- With infinite resources, we get exponential growth, but rabbits breed more quickly.

  e.g. $\dot{x} = 3x \Rightarrow x = x_0 e^{3t}$

  $\dot{y} = 2y \Rightarrow y = y_0 e^{2t}$

- With limited resources, we get the logistic equation with fixed carrying capacity

  e.g. $\dot{x} = x(3-x)$ (with units in millions, so carrying capacity is 3 million rabbits or 2 million sheep)

  $\dot{y} = y(2-y)$

  $\frac{dx}{dt} = x(3-x)$

$$\hookrightarrow \frac{dx}{dt} = x(3-x)$$

$$\frac{dx}{x(3-x)} = dt$$

$$\left[\frac{\frac{1}{3}}{x} + \frac{\frac{1}{3}}{3-x}\right] dx = dt$$

$$\frac{1}{3}\left[\frac{1}{x} + \frac{1}{3-x}\right] dx = dt$$

$$\frac{1}{3}\Big[\ln|x| - \ln|3-x|\Big] = t + C$$

$$\ln\left|\frac{x}{3-x}\right| = 3t + C$$

$$\left|\frac{x}{3-x}\right| = Ce^{3t}$$

$$\left|-1 + \frac{3}{3-x}\right| = Ce^{3t}$$

If $0 < x < 3$,

$$-1 + \frac{3}{3-x} = Ce^{3t}$$

$$\frac{3}{3-x} = 1 + Ce^{3t}$$

$$3 - x = \frac{3}{1+Ce^{3t}}$$

$$x = 3 - \frac{3}{1+Ce^{3t}}, \quad C>0$$

If $x<0$ or $x>3$

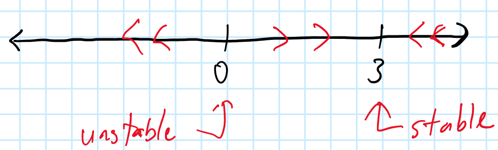
$$-1 + \frac{3}{3-x} = -Ce^{3t}$$

$$\frac{3}{3-x} = 1 - Ce^{3t}$$

$$3 - x = \frac{3}{1-Ce^{3t}}$$

$$x = 3 - \frac{3}{1-Ce^{3t}}, \quad C > 0$$

Or $x=0$ and $x=3$ are also solutions



- But both compete for the same food, so they come into conflict. These conflicts are proportional to both populations. Since sheep are bigger, let's say they win more often.

i.e. $\dot{x} = x(3-x) - 2xy = x(3-x-2y)$

$\dot{y} = y(2-y) - xy = y(2-x-y)$

What is the behavior of this dynamical system? Do the rabbits and sheep live in harmony, or does one win and drive the other extinct?

Let's look for fixed points, i.e., when $\dot{x}=0$ and $\dot{y}=0$.

Let's look for fixed points, i.e., when $\dot{x}=0$ and $\dot{y}=0$.

$$\left.\begin{aligned} 0 &= x(3-x-2y) \\ 0 &= y(2-x-y) \end{aligned}\right\}$$

Case 1: $x=0$. $\Rightarrow 0=y(2-y) \Rightarrow y=0, 2$.

Case 2: $x\neq 0$. $\Rightarrow 0=3-x-2y$

$\Rightarrow x=3-2y$

$0=y[2-(3-2y)-y]$

$=y(-1+y)=0$

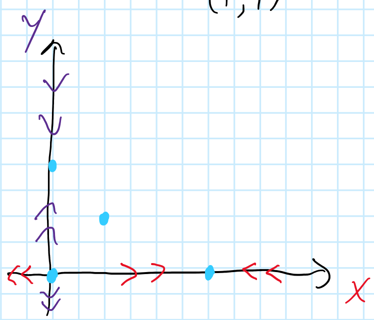
$\Rightarrow y=0, \quad y=1$

If $y=0$, $x=3$

If $y=1$, $x=1$

4 fixed points: $(0,0)$, $(0,2)$, $(3,0)$, $(1,1)$

Can we say anything about the qualitative behavior around these points?



The behavior along the axes precisely matches logistic growth of each of rabbits + sheep separately.

But how do the two populations interact?
Can we talk about stability of each of these fixed points?

Stability: Here, we will not discuss the stability of the system as a whole because there are multiple fixed points.

We call a fixed point x_0 **attracting** if all trajectories starting near x_0 approach it as $t\to\infty$.

We call a fixed point **Liapunov stable** if all trajectories that start sufficiently close to x_0 remain close for all time.

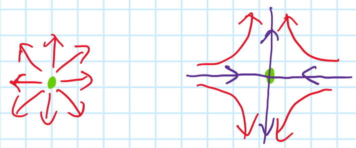
If a fixed point is both **attracting** and **Liapunov stable**, then we say it is **asymptotically stable**.

If a fixed point is both attracting and Liapunov stable, then we say it is asymptotically stable.

If a fixed point is Liapunov stable but not attracting, we call it neutrally stable.

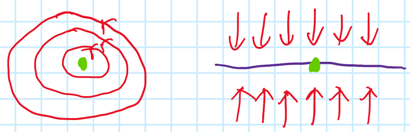
Ex.

Attracting & Liapunov stable



Not attracting

Not Liapunov stable

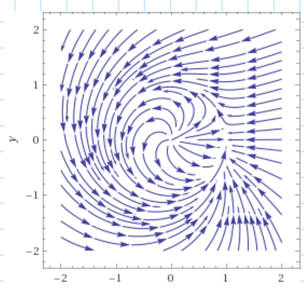


Not attracting

Liapunov stable

$$\dot{x} = x + xy - (x+y)\sqrt{x^2+y^2} \qquad \approx \qquad \dot{r} = r(1-r)$$

$$\dot{y} = y - x^2 + (x-y)\sqrt{x^2+y^2} \qquad\qquad \dot{\varphi} = r(1-\cos\varphi)$$



$(0,1)$ is attracting but not Liapunov stable

(Very rare; normally attractive implies stable)

https://www.wolframalpha.com/input/?i=streamplot%5B%7Bx+%2B+xy+-+%28x%2By%29%28x%5E2%2By%5E2%29%5E0.5%2C+y+-+x%5E2+%2B+%28x-y%29%28x%5E2%2By%5E2%29%5E0.5%7D%2C+%7Bx%2C-2%2C2%7D%2C+%7By%2C-2%2C2%7D%5D

Let's linearize our ODE around a point (x_0, y_0)

$$\dot{x} = x(3-x-2y) \qquad \text{let} \quad z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \dot{z} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}.$$

$$\dot{y} = y(2-x-y)$$

Recall the Taylor series of a function $f(x,y)$ around (x_0, y_0)

$$f(x,y) = f(x_0,y_0) + (x-x_0) f_x(x_0,y_0) + (y-y_0) f_y(x_0,y_0) + \text{higher order terms}$$

$$\dot{z}(x,y) = \begin{pmatrix}\dot{x}(x,y)\\ \dot{y}(x,y)\end{pmatrix} \approx \begin{pmatrix} \dot{x}(x_0,y_0) + (x-x_0)\frac{\partial \dot{x}}{\partial x}(x_0,y_0) + (y-y_0)\frac{\partial \dot{x}}{\partial y}(x_0,y_0) \\ \dot{y}(x_0,y_0) + (x-x_0)\frac{\partial \dot{y}}{\partial x}(x_0,y_0) + (y-y_0)\frac{\partial \dot{y}}{\partial y}(x_0,y_0) \end{pmatrix}$$ for (x,y) close to (x_0,y_0).

Around a fixed point (x_0,y_0), $\dot{x}(x_0,y_0) = \dot{y}(x_0,y_0) = 0$.

$$\dot{z}(x,y) \approx \underbrace{\begin{bmatrix} \frac{\partial \dot{x}}{\partial x}(x_0,y_0) & \frac{\partial \dot{x}}{\partial y}(x_0,y_0) \\ \frac{\partial \dot{y}}{\partial x}(x_0,y_0) & \frac{\partial \dot{y}}{\partial y}(x_0,y_0) \end{bmatrix}}_{\text{Jacobian matrix}} \underbrace{\begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}}_{\text{Translated coordinates with } x_0, y_0 \text{ as the origin.}}$$

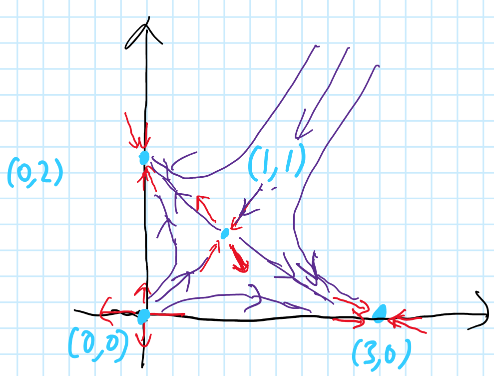
$\Rightarrow$ $\dot{\bar{z}} \approx A\bar{z}$, where $\bar{z} = \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}$.

Thus, around a fixed point, the behavior is approximately that of the linear system with the Jacobian as the matrix.

Back to rabbits and sheep.

$$\dot{x} = x(3-x-2y) = 3x - x^2 - 2xy$$
$$\dot{y} = y(2-x-y) = 2y - xy - y^2$$

$$J(x,y) = \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{bmatrix} = \begin{bmatrix} 3-2x-2y & -2x \\ -y & 2-x-2y \end{bmatrix}$$



$J(0,0) = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow$ $\lambda = 3, 2$ source $\quad v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$J(0,2) = \begin{bmatrix} -1 & 0 \\ -2 & -2 \end{bmatrix} \rightarrow$ $\lambda = -1, -2$ sink $\quad v_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$J(3,0) = \begin{bmatrix} -3 & -6 \\ 0 & -1 \end{bmatrix} \rightarrow$ $\lambda = -1, -3$ sink $\quad v_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$J(1,1) = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix} \rightarrow$ $\lambda^2 + 2\lambda - 1 = 0$
$\lambda = -1 \pm \sqrt{8} = -1 \pm \sqrt{2}$
saddle point

$\lambda_1 = -1+\sqrt{2}$ $\quad \lambda_2 = -1-\sqrt{2}$

$v_1 = \begin{bmatrix} -\sqrt{2} \\ \end{bmatrix}$ $\quad \begin{bmatrix} \sqrt{2} \\ \end{bmatrix}$

Qualitative analysis

(0,2) and (3,0) are stable,

but (1,1)

$(0,2)$ and $(3,0)$ are stable,
but not $(1,1)$.

$\lambda_1 = -1+\sqrt{2}$ $\qquad$ $\lambda_2 = -1-\sqrt{2}$

$v_1 = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$ $\qquad$ $v_2 = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$

So the sheep and rabbits cannot live in harmony except exactly at $(1,1)$, but any perturbation will send it away.

Note: this qualitative analysis only works for non-borderline cases, i.e. when $Re(\lambda_1) \neq 0$ and $Re(\lambda_2) \neq 0$ and $\lambda_1 \neq \lambda_2$.

Ex. $\dot{x} = x^2 - y$
$\dot{y} = x - y$

Fixed pts: $\left.\begin{array}{l} x^2 - y = 0 \\ x - y = 0 \end{array}\right\}$ $\begin{array}{l} x^2 - x = 0 \\ x = y \end{array}$ $\quad$ $\begin{array}{l} x = 0, 1 \\ y = 0, 1 \end{array}$

$$J = \begin{bmatrix} 2x & -1 \\ 1 & -1 \end{bmatrix}$$

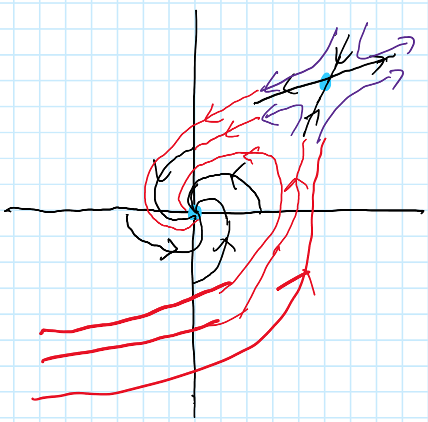
$(0,0) \Rightarrow J = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$

$\lambda^2 + \lambda + 1 = 0$

$\lambda = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

spiral sink $\quad \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

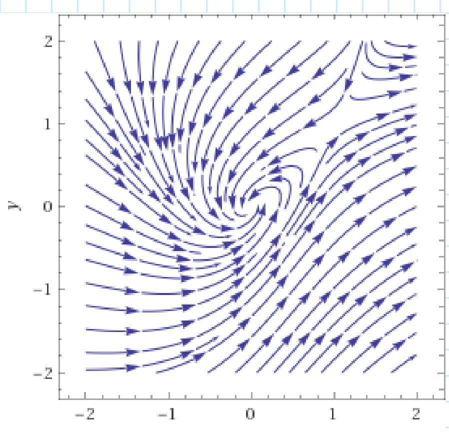
counterclockwise



$(1,1) \Rightarrow J = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$

$\lambda^2 - \lambda - 1 = 0$

$\lambda = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$



$\lambda_1 = \frac{1}{2} + \frac{\sqrt{5}}{2} > 0$ $\qquad$ $\lambda_2 = \frac{1}{2} - \frac{\sqrt{5}}{2} < 0$

$\begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (\frac{1}{2} + \frac{\sqrt{5}}{2}) x_1 \\ (\frac{1}{2} + \frac{\sqrt{5}}{2}) x_2 \end{bmatrix}$ $\qquad$ $\begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (\frac{1}{2} - \frac{\sqrt{5}}{2}) x_1 \\ (\frac{1}{2} - \frac{\sqrt{5}}{2}) x_2 \end{bmatrix}$

$\Rightarrow x_1 - x_2 = \left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right) x_2$ $\qquad$ $x_1 - x_2 = \left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right) x_2$

$x_1 = \left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right) x_2$ $\qquad$ $x_1 = \left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right) x_2$

$v_1 = \begin{bmatrix} \frac{3}{2} + \frac{\sqrt{5}}{2} \\ 1 \end{bmatrix} \approx \begin{bmatrix} 2.6 \\ 1 \end{bmatrix}$ $\qquad$ $v_2 = \begin{bmatrix} \frac{3}{2} - \frac{\sqrt{5}}{2} \\ 1 \end{bmatrix} \approx \begin{bmatrix} 0.4 \\ 1 \end{bmatrix}$

Difference equations

Difference equations

For ODEs, time is taken to be continuous. This is great when the phenomenon is from e.g. physics where there isn't a minimum time. What can we do when time is discrete? e.g. yearly interest from a bank account, or cicadas that reproduce once every 13 years.

Define: A first order difference equation by

$$y_{n+1} = f(n, y_n), \quad n = 0, 1, 2, 3, \ldots, \quad y_n \in \mathbb{R}.$$

It is linear if the function f is linear.

We can give initial conditions $y_0 = \alpha$.

Often, we care about equations of the form $y_{n+1} = f(y_n)$, with no direct dependence on "time" n. i.e. the equivalent of autonomous

Define: An equilibrium solution to be solution that is constant for all y_n. For autonomous difference equations, these correspond to fixed points $y_n = f(y_n)$.

Ex. You have a bank account that gives 10% interest per year, compounded annually. How much money is in your bank account after n years, if you started with \$100.

$$\left.\begin{aligned} y_{n+1} &= 1.1\, y_n \\ y_0 &= 100 \end{aligned}\right\} \quad y_n = (1.1)^n y_0 = (1.1)^n \cdot 100.$$

No equilibrium solution because your money grows indefinitely.

Ex. You have \$26,000 in student debt, which charges 6% (actually 6.45% right now) interest per year, compounded monthly, i.e. each month you are charged 0.5% interest. Each month you pay \$200. How long will it take to pay off your loan?

Let $n = 1, 2, \ldots$ be number of months since graduation.

$$y_{n+1} = (1.005) \cdot y_n - 200, \quad y_0 = 26000$$

Let's solve the more general case

$$y_{n+1} = r \cdot y_n - b$$

$$y_1 = r \cdot y_0 - b$$

$$y_2 = r(r \cdot y_0 - b) - b = r^2 \cdot y_0 - r \cdot b - b$$

$$y_3 = r^3 y_0 - r^2 b - rb - b$$

$$\vdots$$

$$y_n = r^n y_0 - r^{n-1} b - r^{n-2} b - \cdots - rb - b$$

$$y_n = r^n y_0 - b \cdot \sum_{i=1}^{n-1} r^i = r^n y_0 - \frac{1-r^n}{1-r} \cdot b$$

Setting $y_n = 0$ and solving for n,

$$0 = r^n y_0 - \frac{1-r^n}{1-r} \cdot b$$

$$r^n y_0 = \frac{1-r^n}{1-r} \cdot b$$

$$\frac{r^n}{1-r^n} = \frac{1}{1-r} \cdot \frac{b}{y_0}$$

$$-1 + \frac{1}{1-r^n} = \frac{1}{1-r} \cdot \frac{b}{y_0}$$

$$\frac{1}{1-r^n} = 1 + \frac{1}{1-r} \cdot \frac{b}{y_0}$$

$$1 - r^n = \frac{1}{1 + \frac{1}{1-r} \cdot \frac{b}{y_0}}$$

$$r^n = 1 - \frac{1}{1 + \frac{1}{1-r} \cdot \frac{b}{y_0}}$$

$$n \log r = \log\left(1 - \frac{1}{1 + \frac{1}{1-r} \cdot \frac{b}{y_0}}\right)$$

$$n = \log\left(1 - \frac{1}{1 + \frac{1}{1-r} \cdot \frac{b}{y_0}}\right) \Big/ \log r$$

Letting $r = 1.005$, $b = 200$, $y_0 = 26000$

$n = 210.489$ months. ≈ 17.5 years

What if you pay \$300 a month instead?

What if you pay \$300 a month instead?

$n = 113.881$ months ≈ 9.4 years